ISOLDE — A Maple Package for Systems of Linear Functional Equations

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May 29, 2012

Abstract

The Maple ISOLDE package contains algorithms for symbolically solving various classes of linear functional systems. In this demonstration, we focus on the local analysis of systems of linear differential and difference equations. We illustrate the use of our formal reduction algorithm and sketch the link with the computation of a formal fundamental matrix solution at a regular or irregular singularity.

1 Introduction

The ISOLDE[1] package (available at `http://isolde.sourceforge.net`) had been created in the late 90s. It was initially designed for symbolically solving systems of first order linear differential equations. In particular, tasks such as transforming and simplifying the system, classifying its singularities [7], computing normal forms and formal solutions [6, 8] are part of the local analysis functionality provided by ISOLDE. Algorithms for computing several important classes of solutions such as rational and exponential solutions are also implemented in the package. Furthermore, a method for computing factorizations of completely reducible systems [5] is included.

More recently, we started to study algorithms for solving other types of systems such as difference and q-difference systems. Using pseudo-linear algebra, we adopt a unifying view and design generic algorithms that can be applied to individual types of systems by giving specific input parameters [4, 1]. In this note we report on additional functionality of ISOLDE: an extension of the *formal reduction* method from [8] to difference systems.

2 Notations and Basic Definitions

Let F be a field of characteristic 0 and n an integer with $n \geq 2$. A system of linear pseudo-differential equations over F has the form

$$\delta y = A\phi y \tag{1}$$

where $A \in F^{n\times n}$ and y a vector of n unknowns. Here, ϕ is an automorphism of F and δ a *pseudo-derivation* with respect to ϕ, i.e. maps satisfying $\delta(f+g) = \delta f + \delta g$ and $\delta(fg) = \delta f \phi g + f \delta g$ for all $f, g \in F$.

[1] **I**ntegration of **S**ystems of **O**rdinary **L**inear **D**ifferential **E**quations

Given a local parameter $\tau \in F$, we define a τ-adic *valuation* $v : F \longrightarrow \mathbb{Z} \cup \{+\infty\}$ by $v(f) = m$ if $f = \tau^m(f_0 + f_1\tau + \ldots), f_0 \neq 0$ and $v(0) = +\infty$. Furthermore, for a continuous map δ, its *degree* is given by $\omega(\delta) = v\left(\tau^{-1}\delta(\tau)\right)$. This leads to a unique expansion of the system matrix (1) in the form

$$A = \frac{1}{\tau^{p-\omega}} \sum_{i=0}^{\infty} A_i \tau^i, \quad p \geq 0 \tag{2}$$

where the coefficient matrices A_i have entries in $\bar{F}$ the residue field of F with respect to v. Note that $p \in \mathbb{N}$ is the *Poincaré-rank* of the system.

In this presentation we are interested in two specific cases of the system (1). For both cases, we let K be a field of characteristic zero with $\mathbb{Q} \subseteq K$.

(i) Linear differential systems over $F = K((x))$. Here we choose $\tau = x, \phi = \text{id},\ \delta = \frac{d}{dx}$ and $\omega = -1$.

(ii) Linear difference systems over $F = K((x^{-1}))$ equipped with the x^{-1}-adic valuation and $\phi(\tau) = \frac{\tau}{\tau+1},\ \delta = \phi - \text{id},\ \omega = 1$.

3 An Example

In this section, we give an example ISOLDE session, illustrating the formal reduction of linear differential and difference systems. We will use the following rational function matrix

$$A := \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{-1+5\,x^4}{x^4} \\ 0 & \frac{4+2\,x^4}{(x-3/2)^2x^2} & \frac{-1+x^5}{(x-3/2)^3x^2} & 0 & 0 \\ 0 & 0 & \frac{2+5\,x^5}{(x+1/5)(x-3/2)^3x^2} & \frac{-2}{(x-3/2)^2x^2} & 0 \\ \frac{-4\,x^3+3\,x^2}{(x-3/2)x^2} & \frac{4\,x+3\,x^3}{(x-3/2)^3x} & 0 & \frac{-2\,x+2\,x^2}{(x+1/5)(x-3/2)^3x} & 0 \\ 0 & \frac{7}{x(x-3/2)^2} & 0 & 0 & 0 \end{bmatrix}$$

throughout this section. ISOLDE allows the handling of systems with formal power series coefficients, which is implemented using *lazy evaluation*. For this purpose, every matrix is represented by an internal data structure which stores computed coefficients and allows for the computation of new ones. The above matrix can be converted to a series matrix at $x = 0$ using the command

```
> M := mat_convert(A,x,0);
```

$$M := A1$$

The return value is a unique key of a global hash table, which stores the corresponding internal data structure. This key can then be used as input for subsequent function calls, such as the `FormalReduce` function:

```
> fr := FormalReduce(M,x,lambda);
```

$$\left[\left[\frac{16}{9t^2} + \frac{36187}{27648t}, A16\right], \left[-\frac{80}{27t^2} + \frac{9072691}{1152000t}, A20\right], \left[\frac{196}{2025t^5} + \frac{49}{6750t^4} - \frac{439621}{1080000t^3}, A37\right]\right]$$

This command computes the formal reduction of the system (1). By default, ISOLDE uses the maps ϕ, δ and the value of ω that correspond to the differential case (i). The output above is a list of pairs, each of which contains an *exponential part* of the system and a key to a new system with Poincaré-rank zero and certain additional properties. The exponential parts, together with regular solutions of the individual new systems, can be used to construct a fundamental system of formal solutions of the original system. In this example, the last pair actually represents three exponential parts and corresponding systems. The reason for this is that it contains a ramification which can be seen using the command

```
>  mat_get_ramification(fr[3,2]);
```

$$x = \frac{45}{14}\,t^3$$

We will now illustrate how the same function can be used for the formal reduction of difference systems, as well. Let us define new values for the maps ϕ, δ and ω in order to model the system (ii):

```
>  phi:= proc(f, x)  subs(x=x/(x+1),f) end proc;
>  delta:= proc(f, x) f - phi(f,x) end proc;
>  omega:=1;
```

We now call the function `FormalReduce` with the same input matrix, but additional parameters.

```
>  FormalReduce(M,x,lambda, phi, delta, omega);
```

$$\left[\left[-\frac{80}{27t^2}, A54\right], \left[\frac{16}{9t^2}, A60\right], \left[\frac{14}{15t^3}, A79\right]\right]$$

The format of the output remains the same; it now contains the *hyperexponential parts* of the system.

References

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