Bivariate Rank-Reduction Techniques and their Application for Linear Functional Systems

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Notations

- K a field, $char(K) = 0$
- $R = K((y))((x)) \cong K((x, y))$
- $\mathcal{U} = GL_n(R)$ the group of units of $R^{n \times n}$

- v_x a discrete valuation on R s.t. $v_x(0) = \infty$ and $v_x(a) = h$ if $a = \sum_{i \geq h} a_i(y) x^i$ with $a_h \neq 0$
- lc_x the leading coefficient a_h of $a \neq 0$

We can define v_y and lc_y in an analogous way.

Outline

Bivariate algebraic rank reduction

Applications to singular perturbation systems

Some definitions

Definition

For $A \in R^{n \times n}$ define the *Moser-weight* of A in x

$$m_x(A) := -v_x(A) + \frac{\mathrm{rank}\,(\mathrm{lc}_x(A))}{n}.$$

How do $m_x(A), m_y(A)$ change under similarity transformations, i.e. $T^{-1}AT$ with $T \in \mathcal{U}$?

Definition

$T \in \mathcal{U}$ is an *algebraic reducing transformation* w.r.t. x if

$$m_x(T^{-1}AT) < m_x(A).$$

T is called *compatible w.r.t. y* iff

$$m_y(T^{-1}AT) \leq m_y(A).$$

Example

$$A(x,y) = \begin{pmatrix} 0 & x^l y^m \\ x^{l+2} y^{m+2} & 0 \end{pmatrix}, \text{ for } l, m \in \mathbb{Z}$$

with $m_x(A) = -l + 1/2$ and $m_y(A) = -m + 1/2$.

A few transformation matrices:

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}, \quad T_3 = \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}.$$

- T_1 is reducing w.r.t. x, compatible w.r.t. y
- T_2 is reducing w.r.t. y, compatible w.r.t. x
- T_3 is reducing w.r.t. x but *not* compatible w.r.t. y

The Bivariate Algebraic Rank-Reduction Problem

Given $A \in R^{n \times n}$ find its minimal Moser-weight in x using only reducing transformations $T \in \mathcal{U}$ compatible w.r.t. y.

Theorem (Moser, 1960)

Given $A \in R^{n \times n}$ with

$$A(x,y) = x^{v_x(A)} \sum_{i \geq 0} A_i(y) x^i, \ (A_0 \neq 0).$$

There exists an algebraic reducing transformation T w.r.t. x iff

$$\theta_x(\lambda) = \ x^{\text{rank } A_0} det(\lambda I + x^{-v_x(A)+1} A)\Big|_{\lambda=0} \equiv 0.$$

Normalization of $A_0 = \mathrm{lc}_x(A)$

Lemma

Let $A_0 \in K((y))^{n\times n}$ and $r = \mathrm{rank}\ A_0$.
There exists a unimodular matrix $U \in GL_n(K((y)))$ such that

$$U^{-1}AU = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

with $B \in K((y))^{r\times n}$.

The G-matrix (Hilali and Wazner, 1987)

$$A = x^{v_x(A)} \left(A_0 + xA_1 + \mathcal{O}(x^2) \right)$$

Using the lemma we have

$$A_0 = \begin{bmatrix} A_0^{11} & A_0^{12} \\ 0 & 0 \end{bmatrix}$$

where $\begin{bmatrix} A_0^{11} & A_0^{12} \end{bmatrix} \in K((y))^{r \times n}$.
We also decompose A_1 into similar blocks:

$$A_1 = \begin{bmatrix} A_1^{11} & A_1^{12} \\ A_1^{21} & A_1^{22} \end{bmatrix}.$$

The G-matrix of A is defined as

$$G(A, \lambda) = \begin{bmatrix} A_0^{11} & A_0^{12} \\ A_1^{21} & A_1^{22} - \lambda I \end{bmatrix}.$$

One can verify that $\det G(A, \lambda) = \theta(\lambda) \in K((y))[\lambda]$.

Normalization of G-matrix

Lemma

If $\theta(\lambda) \equiv 0$ *there exists a unimodular* $U \in GL_n(K((y)))$ *such that*

$$G(U^{-1}AU, \lambda) = \begin{bmatrix} G^{11} & G^{12} & 0 \\ G^{21} & G^{22} - \lambda & 0 \\ G^{31} & G^{32} & G^{33} - \lambda \end{bmatrix}$$

where $0 \leq q \leq n$ *and*
$G^{11} \in K((y))^{l \times l}, G^{22} \in K((y))^{(n-l-q)\times(n-l-q)}, G^{33} \in K((y))^{q\times q}$.

Additionally,

$$\operatorname{rank} \begin{bmatrix} G^{11} \\ G^{21} \end{bmatrix} + n - q = \operatorname{rank} \begin{bmatrix} G^{11} & G^{12} \\ G^{21} & G^{22} \end{bmatrix}$$

and G^{33} *is lower triangular with a zero-diagonal.*

The reduction step

Proposition

Assume the system's G-matrix has been normalised as in the previous lemma. Then, the diagonal matrix

$$D := diag(xI_l, I_{m-l-q}, xI_q)$$

satisfies

$$m_x(D^{-1}AD) < m_x(A).$$

We also (trivially) have

$$m_y(D^{-1}AD) = m_y(A).$$

MoserReductionStep

Input: $A \in R^{n \times n}, (x, y)$

Output: T such that
$m_x(T^{-1}AT) < m_x(A)$ and $m_y(T^{-1}AT) \leq m_y(A)$

Normalize A_0 using a unimodular transformation U
Normalize G using another unimodular transformation U
Compute D

return $D^{-1}U^{-1}AUD$

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Conclusion

- Use the bivariate rank-reduction as part of a formal reduction algorithm for singular perturbation systems
- Implementation in ISOLDE
- Consider systems of difference and q-difference equations
- What conditions are sufficeint for the multivariate case?