On Recurrences for Ising Integrals

Flavia Stan*
Research Institute for Symbolic Computation (RISC)
Johannes Kepler University Linz, Austria

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Abstract

We use WZ summation methods to compute recurrences for the Ising-class integrals $C_{n,k}$. In this context, we describe an algorithmic approach to obtain homogeneous and inhomogeneous recurrences for a general class of multiple contour integrals of Barnes' type.

1 Introduction

We give an affirmative answer to one of the problems stated in Section 7 of [3] regarding recurrences in $k \geq 1$ for the members of the Ising-class integrals

$$C_{n,k} := \frac{1}{n!} \int_0^\infty \cdots \int_0^\infty \frac{dx_1 dx_2 \dots dx_n}{(\cosh x_1 + \cdots + \cosh x_n)^{k+1}}. \tag{1}$$

In [3], after transforming the member $C_{5,k}$ of the Ising-class integrals to a two-fold nested Barnes integral, it was asked whether, using this new transformed form of the integral, an already conjectured recurrence could be proven by means of computer algebra algorithms based on WZ summation methods. As described in [3], this idea goes back to W. Zudilin.

Using Wegschaider's algorithm [10], which is an extension of multivariate WZ summation [12], we have obtained the conjectured recurrences in $k \geq 1$ for the integrals $C_{5,k}$ and $C_{6,k}$. Moreover, we will show that, in principle, one can obtain recurrences with respect to $k \geq 1$ for any integral of the form (1) with $n \in \mathbb{N}$, using the multivariate summation algorithm [10] after applying the above mentioned transformation method to the integral.

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In [4], J. M. Borwein and B. Salvy show the existence of linear recurrences with polynomial coefficients for the integrals (1) using the theory of D-finite series and a Bessel-kernel representation given in [2]. Also a very efficient algorithm to compute recurrences in $k \geq 1$ for the integrals $C_{n,k}$ for given $n \in \mathbb{N}$ is described in [4]. Our approach to obtain these recurrences is different; in particular, it gives a general algorithmic method to compute homogeneous and inhomogeneous recurrences for multiple nested Barnes' type integrals.

2 The Problem

For the statement of the problem we invoke the renormalization

$$c_{n,k} := \frac{n!}{2^n}\Gamma\left(k+1\right)C_{n,k}$$

used in [3]. The idea of W. Zudilin presented in Section 7 of [3] relies on the following analytic convolution theorem.

Theorem 1 *([3], Theorem 7) For $k \in \mathbb{C}$ with $Re(k) > 0$ and $n, q \in \mathbb{N}$ such that $n \geq 1$ and $1 \leq q \leq n-1$, we have*

$$c_{n,k} = \frac{1}{2\pi i}\int_{\mathbf{C}} c_{n-q,k+s}c_{q,-1-s}ds$$

where the contour $\mathbf{C}$ runs over the vertical line $(-\lambda - i\infty, -\lambda + i\infty)$ with $\lambda \in \mathbb{R}$ such that $-1 - Re(k) < -\lambda < -1$.

Also in [3] the closed forms

$$C_{1,k} = \frac{2^k\Gamma\left(\frac{k+1}{2}\right)^2}{\Gamma\left(k+1\right)} \tag{2}$$

and

$$C_{2,k} = \frac{\sqrt{\pi}\Gamma\left(\frac{k+1}{2}\right)^3}{2\Gamma\left(\frac{k}{2}+1\right)\Gamma\left(k+1\right)} \tag{3}$$

were computed. In the following sections, we will compute the homogeneous recurrences satisfied by $C_{3,k}$ and $C_{6,k}$ for $k \geq 1$. By applying Theorem 1, using the closed forms (2) and (3), and making substitutions $(s,t) \to (2s, 2t)$, we are able to rewrite these as the Barnes' type integrals

$$C_{3,k} = \frac{1}{12i\sqrt{\pi}\Gamma(k+1)}\int_{\mathcal{C}_s}\frac{\Gamma\left(\frac{k+1}{2}+s\right)^3\Gamma(-s)^2}{\Gamma\left(\frac{k}{2}+s+1\right)4^s}ds \tag{4}$$

and, respectively,

$$C_{6,k} = \frac{-1}{720\sqrt{\pi}\Gamma\left(k+1\right)} \int_{\mathcal{C}_s} \int_{\mathcal{C}_t} \frac{\Gamma\left(\frac{k+1}{2}+s\right)^3 \Gamma\left(t-s\right)^3 \Gamma\left(-t\right)^3}{\Gamma\left(\frac{k}{2}+s+1\right)\Gamma\left(t-s+\frac{1}{2}\right)\Gamma\left(-t+\frac{1}{2}\right)} ds dt. \tag{5}$$

The vertical contours $\mathcal{C}_s := (-\lambda - i\infty, -\lambda + i\infty)$ separate the poles of $\Gamma\left(\frac{k+1}{2}+s\right)$ from the poles of $\Gamma\left(-s\right)$ and, respectively, from those of $\Gamma\left(t-s\right)$. Similarly, $\mathcal{C}_t := (-\rho - i\infty, -\rho + i\infty)$ splits the descending set of poles coming from $\Gamma\left(t-s\right)$ from the ascending poles of $\Gamma\left(-t\right)$. For reasons that become clear in Section 5, we choose $\lambda, \rho \in \mathbb{R}$ such that the following conditions are satisfied:

$$-\frac{1+Re(k)}{2} < -\lambda < -\rho < -1. \tag{6}$$

Successively applying Theorem 1, we prove the following integral representation:

Proposition 1 *For arbitrary integers $n, k \geq 1$, we have*

$$C_{n,k} = \frac{2^n}{n!\left(2\pi i\right)^q} \frac{1}{\Gamma\left(k+1\right)} \int_{\mathbf{C}_{t_1}} \cdots \int_{\mathbf{C}_{t_q}} c_{2,k+t_1} \left(\prod_{j=1}^{q-1} c_{2,-1-t_j+t_{j+1}}\right) c_{\epsilon,-1-t_q} dt_1 \ldots dt_q, \tag{7}$$

where $q := \left\lceil \frac{n}{2} \right\rceil - 1$ and $\epsilon := n - 2q$.

We use the closed forms (2) and (3), and the substitutions $t_j \to 2t_j$ for all $1 \leq j \leq q$, to obtain from (7) the final representation of $C_{n,k}$ for arbitrary $k, n \geq 1$. At last, we choose new integration contours $\mathcal{C}_{t_j} := (-\lambda_j - i\infty, -\lambda_j + i\infty)$ for all $1 \leq j \leq q$ which run over vertical lines separating the poles of gamma functions of the form $\Gamma\left(a+t_j\right)$ from the poles of gamma functions of the form $\Gamma\left(b-t_j\right)$. For reasons presented later, we choose these Barnes paths of integration such that the following conditions are satisfied:

$$-\frac{1+Re(k)}{2} < -\lambda_1 < -\lambda_2 < \cdots < -\lambda_q < -1. \tag{8}$$

3 Deriving Recurrences Algorithmically

Wegschaider's algorithm [10] is an extension of multivariate WZ summation [12], and in this context it is used to compute recurrences for sums of the form

$$Sum\left(\mu\right) = \sum_{\kappa_1 \in \mathcal{R}_1} \cdots \sum_{\kappa_r \in \mathcal{R}_r} \mathcal{F}\left(\mu, \kappa_1, \ldots, \kappa_r\right). \tag{9}$$

Under some mild side conditions described in [10], it can be applied if the summands $\mathcal{F}(\mu,\kappa)$ are hypergeometric in all integer variables μ_i from $\mu=(\mu_1,\dots,\mu_p)$ and in all summation variables κ_j from $\kappa=(\kappa_1,\dots,\kappa_r)\in\mathcal{R}$ where $\mathcal{R}:=\mathcal{R}_1\times\cdots\times\mathcal{R}_r\subseteq\mathbb{Z}^r$ is the summation range.

Remark: Recall that an expression $\mathcal{F}(\mu,\kappa)$ is called hypergeometric [13, 12] if there exists a rational function $r_{m,k}(\mu,\kappa)$ such that $\frac{\mathcal{F}(\mu,\kappa)}{\mathcal{F}(\mu+m,\kappa+k)}=r_{m,k}(\mu,\kappa)$ at the points $m\in\mathbb{Z}^p$ and $k\in\mathbb{Z}^r$ where this ratio is defined.

As described in [12], WZ-summation is based on Sister Celine's method [7] of finding a κ-free recurrence for the hypergeometric summand $\mathcal{F}(\mu,\kappa)$

$$\sum_{(u,v)\in\mathbb{S}} c_{u,v}(\mu)\,\mathcal{F}(\mu+u,\kappa+v)=0 \tag{10}$$

where the set of shifts $\mathbb{S}\subset\mathbb{Z}^{p+r}$ is called the structure set of the recurrence.

Denoting by $M=(M_1,\dots,M_p)$ and $K=(K_1,\dots,K_r)$ the forward-shift operators with respect to the variables from μ and respectively from κ and using the multi-index notation, the left hand side of (10) can be viewed as applying to $\mathcal{F}$ the operator

$$P(\mu,M,K):=\sum_{(u,v)\in\mathbb{S}} c_{u,v}(\mu)\,M^uK^v.$$

The next step consists of successively dividing the polynomial recurrence operator P by all forward-shift difference operators

$$\Delta_{\kappa_j}\mathcal{F}(\mu,\kappa):=(K_i-1)\,\mathcal{F}(\mu,\kappa)=\mathcal{F}(\mu,\kappa_1,\dots,\kappa_j+1,\dots,\kappa_r)-\mathcal{F}(\mu,\kappa)$$

to obtain an operator free of shifts in the summation variables κ_j from $\kappa=(\kappa_1,\dots,\kappa_r)$, called the principal part of the recurrence (10).

Wegschaider's algorithm [10] generalizes and optimizes this approach in several directions, but thinking on these simple lines, given a structure set $\mathbb{S}$ together with the hypergeometric summand $\mathcal{F}(\mu,\kappa)$, it computes a certificate recurrence of the form

$$\sum_{m\in\mathbb{S}'} a_m(\mu)\,\mathcal{F}(\mu+m,\kappa)=\sum_{j=1}^{r}\Delta_{\kappa_j}\left(\sum_{(m,k)\in\mathbb{S}_j} b_{m,k}(\mu,\kappa)\,\mathcal{F}(\mu+m,\kappa+k)\right), \tag{11}$$

where the polynomials $a_m(\mu)$, not all zero, and $b_{m,k}(\mu,\kappa)$, as well as the sets of shifts $\mathbb{S}_j\subset\mathbb{Z}^{p+r}$ and $\mathbb{S}'\subset\mathbb{Z}^p$ are determined algorithmically.

Note that the left side of (11) constitutes the principal part of (10). Moreover, since the right side contains the quotients of the successive divisions by the delta operators, each expression inside a delta-part Δ_{κ_j} will be free of shifts in the summation variables κ_i with $1\le i<j$.

Note also that Wegschaider's algorithm [10] finds a certificate recurrence for the hypergeometric term $\mathcal{F}(\mu, \kappa)$, if such a recurrence exists. To be more precise, the algorithm [10] terminates successfully, for a large enough structure set, if we restrict our input class to *proper* hypergeometric summands; see [12] for the definition of proper hypergeometric terms and also regarding the existence conditions for certificate recurrences.

Remark: The right hand side of (11) can always be rewritten as

$$\sum_{j=1}^{r} \Delta_{\kappa_j} \left(\sum_{(m,k)\in\mathbb{S}_j} b_{m,k}(\mu,\kappa)\,\mathcal{F}(\mu+m,\kappa+k) \right) = \sum_{j=1}^{r} \Delta_{\kappa_j} \left(r_j(\mu,\kappa)\,\mathcal{F}(\mu,\kappa) \right), \quad (12)$$

where r_j are rational functions of all variables from $\mu = (\mu_1, \ldots, \mu_p)$ and $\kappa = (\kappa_1, \ldots, \kappa_r)$.

Remark: In the certificate recurrence (11), the coefficients $a_m(\mu)$ are polynomials free of the summation variables κ_j from κ, while the coefficients $b_{m,k}(\mu,\kappa)$ of the delta-parts are polynomials in all the variables from μ and κ.

Finally, the recurrence for the multisum (9) is obtained by summing the certificate recurrence (11) over all variables from κ in the given summation range $\mathcal{R}$. Since it can be easily checked whether the summand $\mathcal{F}(\mu,\kappa)$ indeed satisfies the recurrence (11), the certificate recurrence also provides a proof of the recurrence for the multisum $Sum(\mu)$.

Two further remarks are in place. Since in non-elementary applications finding a κ-free recurrence can be a time and space consuming problem, Wegschaider's algorithm [10] is used after making an Ansatz about the structure set $\mathbb{S}$ of this recurrence. A procedure based on modular computation to obtain a candidate structure set was already used in [5] and it is implemented in the Mathematica package `MultiSum`[1]. This package which includes an implementation of Wegschaider's algorithm [10] can be loaded within a Mathematica session by

In[1]:= **<< MultiSum.m**

MultiSum Package by Kurt Wegschaider (enhanced by Axel Riese and Burkhard Zimmermann) – © RISC Linz – V2.02β (02/21/05)

Secondly, we remark that in many applications the function $\mathcal{F}(\mu,\kappa)$ has a finite support. In these cases, if we sum the recurrence (11) over a domain that is larger then the support of the function, the Δ-parts on the right hand side telescope and the values that are not in the support vanish. So, from the summand recurrence one obtains a homogeneous recurrence for the sum

$$\sum_{m\in\mathbb{S}} a_m(\mu)\, Sum(\mu+m) = 0. \quad (13)$$

[1]available at `http://www.risc.uni-linz.ac.at/research/combinat/software/`

This is not the case in general; i.e., in specific situations human inspection is still needed to pass from the recurrence (11) to a homogeneous or inhomogeneous recurrence for the sum (9). More information on this subject can be found in [12].

4 From Summation to Integration

In this section we will show how Wegschaider's algorithm [10] can be used to determine recurrences for multiple contour integrals of Barnes' type

$$Int\,(\mu) = \int_{\mathcal{C}_{\kappa_1}} \dots \int_{\mathcal{C}_{\kappa_r}} \mathcal{F}\,(\mu, \kappa_1, \dots, \kappa_r)\, d\kappa_1 \dots d\kappa_r, \tag{14}$$

where the integrands $\mathcal{F}\,(\mu, \kappa)$ are proper hypergeometric in all integer variables μ_i from $\mu = (\mu_1, \dots, \mu_p)$ and in all integration variables κ_j from $\kappa = (\kappa_1, \dots, \kappa_r) \in \mathbb{C}^r$.

For instance, the integral representations obtained in Section 2 for $C_{n,k}$ for any $n, k \geq 1$ are of the considered form (14) if we distinguish between the even and odd values of $k \in \mathbb{N}$.

As in the case of the summation problem (9), the fundamental theorem of hypergeometric summation stated by Wilf and Zeilberger in [12] proves the existence of non-trivial certificate recurrences of the form (11) for the function $\mathcal{F}\,(\mu, \kappa)$. Using WZ summation methods, Wegschaider's algorithm [10] delivers recurrences of the form (11) for the hypergeometric integrand from (14). As remarked in Section 3, the coefficients on the left hand side of this recurrence are free of all integration variables $\kappa = (\kappa_1, \dots, \kappa_r)$.

Moreover, although discrete functions are our main interest, one can evaluate the function $\mathcal{F}\,(\mu, \kappa)$ also for complex values of the variables μ_i and κ_j for all $1 \leq i \leq p$ and $1 \leq j \leq r$ except at certain points. In our case, the singularities of the numerator gamma functions need to be excluded from the evaluation domain. The function $\mathcal{F}\,(\mu, \kappa)$ is then continuous on its evaluation domain and by taking limits it can be shown that the computed recurrences (11) hold in $\mathbb{C}^{p+r}$.

Therefore, after successively integrating over the Barnes paths of integration $\mathcal{C}_{\kappa_j}$ for $1 \leq j \leq r$, (11) leads, in some cases, to a homogeneous recurrence for the integration problem (14), i.e.,

$$\sum_{m \in \mathbb{S}} a_m\,(\mu)\, Int\,(\mu + m) = 0. \tag{15}$$

However, again in analogy to the summation case, after integrating over the contours of integration $\mathcal{C}_{\kappa_j}$ for $1 \leq j \leq r$, it is not clear, in general, that we obtain a homogeneous equation of the type (15). Consequently, one needs to analyze the behavior of the contour integrals over the left hand side of (11).

For this purpose, we study the following integration problems:

$$I_j := \int_{\mathcal{C}_{\kappa_j}} \Delta_{\kappa_j} \mathcal{F}(\mu, \kappa)\, d\kappa_j = \int_{\mathcal{C}'_{\kappa_j}} \mathcal{F}(\mu, \kappa)\, d\kappa_j - \int_{\mathcal{C}_{\kappa_j}} \mathcal{F}(\mu, \kappa)\, d\kappa_j, \tag{16}$$

where the Barnes path $\mathcal{C}_{\kappa_j}$ runs vertically over $(c_j - i\infty, c_j + i\infty)$ while $\mathcal{C}'_{\kappa_j}$ denotes the shifted path $(1 + c_j - i\infty, 1 + c_j + i\infty)$ for all $1 \le j \le r$.

For any $1 \le j \le r$, consider now the contour integral I_j^N over a rectangle with vertices at the points $c_j - iN$, $c_j + iN$, $1 + c_j + iN$ and $1 + c_j - iN$ with $N \in \mathbb{N}$; i.e.,

$$\begin{aligned} I_j^N = & \int_{1+c_j-iN}^{1+c_j+iN} \mathcal{F}(\mu, \kappa)\, d\kappa_j + \int_{1+c_j+iN}^{c_j+iN} \mathcal{F}(\mu, \kappa)\, d\kappa_j \\ & + \int_{c_j+iN}^{c_j-iN} \mathcal{F}(\mu, \kappa)\, d\kappa_j + \int_{c_j-iN}^{1+c_j-iN} \mathcal{F}(\mu, \kappa)\, d\kappa_j. \end{aligned} \tag{17}$$

If in any such rectangular region of integration, we have the asymptotic behavior

$$\mathcal{F}(\mu, \kappa) = \mathcal{O}\left(e^{-c|\kappa_j|}\right) \quad \text{as} \quad |\kappa_j| \to \infty \quad \text{with} \quad c > 0, \tag{18}$$

then $I_j^N \to I_j$ as $N \to \infty$. Since the function $\mathcal{F}(\mu, \kappa)$ is dominated by an exponential with negative exponent, it suffices to analyze the integrals (16) instead of the integrals over the right hand side of (12).

On the other hand, we can calculate the integrals (17) by considering the residues of the function $\mathcal{F}(\mu, \kappa)$ at the poles lying inside the closed rectangular contours. Therefore, if for all $1 \le j \le r$, the Barnes paths of integration $\mathcal{C}_{\kappa_j}$ can be chosen such that the function $\mathcal{F}(\mu, \kappa)$ has no poles inside these rectangular regions, then the integrals (16) will be zero. This is why conditions (8) are imposed on the integral representation of $C_{n,k}$ for $n, k \ge 1$.

Under these restrictions, we obtain from the certificate recurrence (11) a homogeneous recurrence (15) for the multiple Barnes' type integral (14). Note that, by a different choice of the integration contours, this method will lead to inhomogeneous recurrences for multiple Barnes integrals which satisfy the asymptotic condition (18).

5 Recurrences for the Integrals $C_{n,k}$

After distinguishing between odd and even values of the parameter k, for an arbitrary Ising-class integral $C_{n,k}$, $n, k \ge 1$, one obtains two representations of the form

$$C_{n,\mu} = \frac{2^{n+2q}}{n!\,(2\pi i)^q} \frac{1}{\Gamma(\mu + 1)} \int_{\mathcal{C}_{t_1}} \cdots \int_{\mathcal{C}_{t_q}} \Psi(\mu, t_1, \ldots, t_q)\, dt_1 \ldots dt_q, \tag{19}$$

where $\mu = \frac{k}{2}$, respectively, $\mu = \frac{k-1}{2}$ such that $\mu \in \mathbb{N}$. In both cases the integrand $\Psi(\mu, \mathbf{t})$ is proper hypergeometric in $\mu \geq 0$ and in all integration variables t_j from $\mathbf{t} = (t_1, \ldots, t_q)$.

Therefore, Wegschaider's algorithm [10] can be applied to deliver a certificate recurrence of the form (11), that can always be rewritten as

$$\sum_{m \in \mathbb{S}} a_m(\mu)\, \Psi(\mu + m, \mathbf{t}) = \sum_{j=1}^{r} \Delta_{t_j} \left(\sum_{(m,\tau) \in \mathbb{S}_j} b_{m,\tau}(\mu, \mathbf{t})\, \Psi(\mu + m, \mathbf{t} + \tau) \right), \qquad (20)$$

where $\mathbb{S}$ is a pre-computed structure set, where $b_{m,\tau}(\mu, \mathbf{t})$ and the coefficients $a_m(\mu)$ are polynomials, the latter free of the integration variables and not all zero. Next we discuss when the recurrence relation, obtained after integrating over the certificate (20), is homogeneous.

Proposition 2 *If the integration contours* $\mathcal{C}_{t_j} := (-\lambda_j - i\infty, -\lambda_j + i\infty)$ *satisfy the conditions (8) and the sets of shifts* $\mathbb{S}_j$ *are of the form*

$$\mathbb{S}_j = \left\{ (m, \tau) \in \mathbb{Z}^{q+1} : m \geq 0, \tau_j < \tau_{j+1} \text{ and } \tau_i = 0 \text{ for } 1 \leq i < j \right\},$$

for $1 \leq j < q$ *and*

$$\mathbb{S}_q = \left\{ (m, \tau) \in \mathbb{Z}^{q+1} : m \geq 0 \text{ and } \tau_i = 0 \text{ for } 1 \leq i \leq q \right\},$$

then we have

$$\int_{\mathcal{C}_{t_j}} \Delta_{t_j} \left(\sum_{(m,\tau) \in \mathbb{S}_j} b_{m,\tau}(\mu, \mathbf{t})\, \Psi(\mu + m, \mathbf{t} + \tau) \right) dt_j = 0, \qquad (21)$$

for all $1 \leq j \leq q$ *and* $\mu \geq 1$.

Proof: Given the iterative construction of the integral representation (7), computed in Section 2, it suffices to study the behavior of the following two integrals

$$I_1 := \int_{\mathcal{C}_1} \Delta_t \left(\frac{\Gamma(t-r)^3}{\Gamma\left(t - r + \frac{1}{2}\right)} \frac{\Gamma(-t)^2}{4^t} \right) dt, \qquad (22)$$

$$I_2 := \int_{\mathcal{C}_2} \Delta_t \left(\frac{\Gamma(t-r)^3}{\Gamma\left(t - r + \frac{1}{2}\right)} \frac{\Gamma(s+1-t)^3}{\Gamma\left(s - t + \frac{3}{2}\right)} \right) dt, \qquad (23)$$

where $r, s \in \mathbb{C}$ are given constants.

We will prove here that both integrals

$$I_l = \int_{\mathcal{C}_l} \Delta_t \left(F_l(t)\right) dt, \quad l \in \{1,2\}$$

are zero if the contours of integration $\mathcal{C}_l$ are the vertical lines $(-\rho_l - i\infty, -\rho_l + i\infty)$ separating the increasing from the decreasing sequences of poles of the gamma functions appearing in the numerators of the integrands $F_l(t)$. The Barnes paths of integration $\mathcal{C}_l$ also fulfill the conditions (8); i.e., $Re(r) < -\rho_l < Re(s) < -1$ for $l \in \{1,2\}$.

Using the transformation $t+1 \to t$ we can write the integrals (22) and (23) as

$$I_l = \int_{\mathcal{C}'_l} F_l\left(t\right) dt - \int_{\mathcal{C}_l} F_l\left(t\right) dt, \quad l \in \{1,2\}$$

where the shifted contours $\mathcal{C}'_l$ run vertically on the line $(1-\rho_l - i\infty, 1-\rho_l + i\infty)$.

Next we define integrals of the form (17),

$$I_l^N := \int\limits_{1-\rho_l - iN}^{1-\rho_l+iN} F_l\left(t\right) dt + \int\limits_{1-\rho_l+iN}^{-\rho_l+iN} F_l\left(t\right) dt + \int\limits_{-\rho_l+iN}^{-\rho_l-iN} F_l\left(t\right) dt + \int\limits_{-\rho_l-iN}^{1-\rho_l-iN} F_l\left(t\right) dt,$$

for $N > 0$ an arbitrary integer and $l \in \{1,2\}$.

Since conditions (8) are fulfilled, there are no poles of the functions $F_l\left(t\right)$ within these closed rectangular contours of integration. Therefore I_l^N are zero for any integer $N \in \mathbb{N}$. It only remains to show that $I_l^N \to I_l$ as $N \to \infty$ for $l \in \{1,2\}$. For this we need to prove that the integrals

$$J_l^N := \int\limits_{1-\rho_l+iN}^{-\rho_l+iN} F_l\left(t\right) dt \quad \text{and} \quad L_l^N := \int\limits_{-\rho_l-iN}^{1-\rho_l-iN} F_l\left(t\right) dt$$

tend to zero as $N \to \infty$.

The following asymptotic representation of the function $\log\Gamma(z)$ for large $|z|$ in the region where $|\arg(z)| \leq \pi - \delta$ and $|\arg(z+a)| \leq \pi - \delta$ with $\delta > 0$,

$$\log\Gamma(z+a) = (z + a - \frac{1}{2}) \log z - z + \mathcal{O}(1) \tag{24}$$

can be found in ([11], 13.6). Using (24) one obtains when $|t| \to \infty$ and $|\arg(t)| < \pi$,

$$F_l\left(t\right) = \mathcal{O}\left(e^{\mathrm{Im}(t)[\arg(-t) - \arg(t)]}\right), \quad l \in \{1,2\}.$$

Here we distinguish two cases, either $\mathrm{Im}(t) > 0$ or $\mathrm{Im}(t) < 0$, and in any of these cases the functions $F_l\,(t)$ fulfill the condition (18) which assures that the integrals J_l^N and L_l^N tend to zero as $N \to \infty$ for $l \in \{1, 2\}$.

Remark: The conditions of our proposition are very restrictive; especially the condition imposed on the set of shifts appearing inside the last delta part Δ_{t_q} rarely occurs in practice. For example, for the integral $C_{3,k}$ in the case $k = 2K$ we use the integral form (4) and the following commands, from the package `MultiSum`, to compute a certificate recurrence and shift it accordingly

In[2]:= $\mathrm{F}\,[\boldsymbol{k\_}, \boldsymbol{s\_}] := \dfrac{\boldsymbol{\Gamma}\left[\frac{k+1}{2} + s\right]^3 \boldsymbol{\Gamma}[-s]^2}{12 i \sqrt{\pi}\boldsymbol{\Gamma}[k+1]\boldsymbol{\Gamma}\left[\frac{k}{2} + s + 1\right] 4^s};$

In[3]:= $\mathbf{FindRecurrence}\,[F[2K, s], K, s, 1]\,;$

In[4]:= $rec = \mathbf{ShiftRecurrence}\,[\%[[1]], \{K, 2\}, \{s, 2\}]$

Out[4]= $-(2K+1)^3 F[K,s] + 4(K+1)(20K^2+40K+21)F[K+1,s] - 36(K+1)(K+2)(2K+3)F[K+2,s] = \Delta_s[(2K+1)^3 F[K,s] + (2K+1)^3 F[K,s+1] - 4(K+1)(20K^2+40K+21)F[K+1,s] - 16(K+1)(2K^2-4sK-K-5s-4)F[K+1,s+1] + 48(K+1)(K+2)(2K+3)F[K+2,s]].$

After integrating both sides of this certificate recurrence with respect to the variable s, we can apply Proposition 2 only to some of the terms appearing inside the delta part. At last, on the remaning terms, Cauchy's residue theorem and the asymptotic property (18) will be used to evaluate the left-over contour integrals occuring on the right hand side of the recurrence. In this way, after computing the following two Mellin-Barnes integrals

$$\begin{aligned}\int_{\mathcal{C}_s} \Delta_s[(2K+1)^3 F[K, s+1] - 16(K+1)(2K^2 - 4sK - K - 5s - 4)\\ F[K+1, s+1]]ds = (2K+1)^3 2\pi i \operatorname*{Res}_{s=-1} F[K, s+1] -\\ 16(K+1)(2K^2 + 3K + 1) 2\pi i \operatorname*{Res}_{s=-1} F[K+1, s+1],\end{aligned}$$

the final recurrence satisfied by $C_{3,2K}$ turns out to be homogeneous. In more general situations, the necessary residue computations tend to be involved but packages such as `Sigma` [8] and `HarmonicSums` [1] can algorithmically simplify the resulting expressions.

6 The Recurrence for the Integral $C_{6,k}$

In [3], the following recurrence for the integral $C_{6,k}$ was conjectured

$$\begin{aligned}&-(k+1)^6 C_{6,k} + 8(k+2)^2(7k^4+56k^3+182k^2+280k+171)\\ &C_{6,k+2} - 16(k+2)(k+3)^2(k+4)(49k^2+294k+500)C_{6,k+4}\\ &+2304(k+2)(k+3)(k+4)^2(k+5)(k+6)C_{6,k+6} = 0 \end{aligned} \tag{25}$$

To prove that the integral (1) for $n = 6$ satisfies the above recurrence, we use the representation (5). First, we input in Mathematica its integrand as a function of $k \geq 0$ and complex variables s and t

In[5]:= $F[k\_, s\_, t\_] := \frac{-\left(\Gamma\left[\frac{k+1}{2}+s\right]\Gamma[t-s]\Gamma[-t]\right)^3}{720\sqrt{\pi}\Gamma[k+1]\Gamma\left[\frac{k}{2}+s+1\right]\Gamma\left[t-s+\frac{1}{2}\right]\Gamma\left[-t+\frac{1}{2}\right]}.$

In the first part of the proof we want to apply Wegschaider's algorithm [10] which was already introduced in Section 3, to obtain a certificate recurrence for $F[k, s, t]$. For this we need the function to be *proper* hypergeometric not only with respect to the integration variables s, t but also with respect to the additional parameter k. This leads to a case distinction between even and odd values of k. In each of the two cases, we introduce a new variable $K \geq 0$ such that the setting of (14) applies. In this way, $C_{6,k}$ can be expressed as a double Barnes type integral over a proper hypergeometric term

$$C_{6,2K+\epsilon} = \frac{-1}{720\pi} \int_{\mathcal{C}_s} \int_{\mathcal{C}_t} \mathcal{F}(K, s, t) ds dt,$$

with $K \geq 0$, $\epsilon \in \{0, 1\}$ and integration contours satisfying the condition (6).

As already pointed out, one can reduce the running time of the summation algorithm [10] by first making an Ansatz for a small structure set of the recurrence. For example, before computing a recurrence relation for $F[2K, s, t]$, we find a structure set with the command

In[6]:= **FindStructureSet [*F*[2*K*, *s*, *t*], *K*, {*s*, *t*} , {2, 2} , 1]**

which gives us two candidates. Using the first candidate we already succeed in finding a certificate recurrence which can be shifted accordingly to obtain a relation of the form (20), i.e.,

In[7]:= **FindRecurrence [*F*[2*K*, *s*, *t*], *K*, {*s*, *t*} , %[[1]], 1, WZ → True] ;**

In[8]:= ***rec* = ShiftRecurrence [%[[1]], {*K*, 3} , {*s*, 2} , {*t*, 1}] .**

The sets of shifts appearing inside the delta parts, Δ_s and Δ_t can be inspected by using the simple Mathematica commands

In[9]:= **Cases [*rec*[[2, 1]], *F*[__], Infinity]**

Out[9]= $\{F[K,s,1+t],F[K,1+s,1+t],F[K,2+s,1+t],F[1+K,s,t],F[1+K,s,1+t],F[1+K,1+s,t],F[1+K,1+s,1+t],F[1+K,2+s,1+t],F[2+K,s,t],F[2+K,s,1+t],F[2+K,1+s,t],F[2+K,1+s,1+t],F[2+K,2+s,1+t],F[3+K,s,t],F[3+K,s,1+t],F[3+K,1+s,t],F[3+K,1+s,1+t]\}$

and

In[10]:= **Cases [*rec*[[2, 2]], *F*[__],** Infinity]

Out[10]= $\{F[K,s,t],F[1+K,s,t],F[2+K,s,t],F[3+K,s,t]\}.$

When integrating with respect to the variables s and t over this certificate recurrence, the conditions of Proposition 2 are fullfilled by the set of shifts appearing in the Δ_t-part and by a subset of the set shifts contained in Δ_s. At last, we evaluate the remaining contour integrals and again we obtain a homogeneous recurrence satisfied by $\mathrm{INT}[K] := C_{6,2K}$. This is returned by the command

In[11]:= **SumCertificate [*rec*] /.SUM → INT**

Out[11]= $(1+2K)^6\mathrm{INT}[K]-32(1+K)^2(171+560K+728K^2+448K^3+112K^4)\mathrm{INT}[1+K]+256(1+K)(2+K)(3+2K)^2(125+147K+49K^2)\mathrm{INT}[2+K]-36864(1+K)(2+K)^2(3+K)(3+2K)(5+2K)\mathrm{INT}[3+K]=0.$

Similarly, in the case $k = 2K + 1$ and $K \geq 0$ the computed recurrence is

Out[11]= $(1+K)^6\mathrm{INT}[K]-(3+2K)^2(87+210K+196K^2+84K^3+14K^4)\mathrm{INT}[1+K]+(2+K)^2(3+2K)(5+2K)(843+784K+196K^2)\mathrm{INT}[2+K]-144(2+K)(3+K)(3+2K)(5+2K)^2(7+2K)\mathrm{INT}[3+K]=0,$

where $\mathrm{INT}[K]$ now denotes $C_{6,2K+1}$.

The last step of the proof consists in obtaining the recurrence for the sequence of integrals $C_{6,k}$ with $k \geq 0$. To this end, we utilize the fact that the sequences $C_{6,2K}$ and $C_{6,2K+1}$ defined for all $K \geq 0$ are P-recursive (also called holonomic [9, 14]); i.e., they satisfy linear recurrences with polynomial coefficients. To compute the desired recurrence, we load, for instance, the Mathematica package

In[12]:= **<< GeneratingFunctions.m**

GeneratingFunctions Package by Christian Mallinger – © RISC Linz – V 0.68 (07/17/03)

From this package, the command REInterlace computes a recurrence that is satisfied by the sequence obtained by interlacing the input recurrences (see [6] for more details). This means, we input the recurrence relations satisfied by $(C_{6,2K})_{K\geq 0}$ and

$(C_{6,2K+1})_{K\geq 0}$, respectively, and obtain a polynomial recurrence for the sequence $C_{6,k}$ with $k \geq 0$. The computed recurrence is exactly (25) and herewith the proof is complete.

7 Conclusion

This algorithmic method to prove and compute recurrences for members of the Ising-class integrals using Wegschaider's algorithm [10] delivers the recurrence conjectured in [3] for $C_{5,k}$ in completely analogous manner. Neglecting practical issues like computation time, this method applies to all $n \geq 1$.

Though, we need to remark that the algorithm [10] determines recurrences, after making an Ansatz about their structure set (i.e., fixing the set of shifts that they contain), by solving a large system of equations over a field of rational functions. Therefore, if the input of the algorithm is too involved, computations might become time consuming.

Basic ingredients of the approach are the representation of the Ising integrals $C_{n,k}$ for $k, n \geq 1$ as nested Barnes' type integrals and the convolution theorem stated in Section 7 of [3], ideas going back W. Zudilin.

In addition, the method briefly explained in Section 4 of this paper is more general and has a wider range of applications that deserve to be explored further.

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