Symbolic Summation for Feynman Integrals

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Joint Mathematics Meetings of the AMS and MAA
Session on Experimental Mathematics
Washigton DC, 5-8 Jan 2009

Symbolic Summation and Perturbative Quantum Field Theory

- main goal of the project:

  algorithms for computing Feynman parameter integrals

- project coordinator:

  Carsten Schneider

- project partners:

  Peter Paule —— Johannes Blümlein

  RISC DESY

A simple example of a Feynman parameter integral:

$$I_f = \sum_{j=0}^{N-2} \int_0^1 \dots \int_0^1 dx_1 \dots dx_4$$

$$\frac{x_1(x_1x_3 - x_2x_3x_1 + x_4 - x_1x_4 + x_4x_1x_2)^{N-j-2}(x_4 - x_1x_4 + x_1x_3)^j}{(1 - x_1x_2)^{1-\epsilon}(1 - x_1)^{\frac{1}{2}\epsilon}(1 - x_2)^{\frac{1}{2}\epsilon}x_2^{-\frac{1}{2}\epsilon}}$$

Strategies to approach Feynman Integrals

- representations in terms of hypergeometric series

$$
{}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}
$$

$$
= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a} dt
$$

- Mellin-Barnes representations

$$
\frac{1}{(A+B)^k} = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\Gamma(-s)\Gamma(k+s)}{\Gamma(k)} A^{-k-s} B^s ds
$$

Remark: the Beta function is also useful

$$
\mathcal{B}(p, q) := \int_0^1 t^{p-1}(1-t)^{q-1} dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.
$$

I_f becomes

$$I_f = \sum_{j=0}^{N-2} \sum_{k=0}^{j} \sum_{l=0}^{N-2-j} \binom{N-2-j}{l} \binom{j}{k} \left(\frac{\Gamma(k+l+2)}{\Gamma(N+1)} + \frac{(-1)^{k+l}}{N\Gamma(N-1-k-l)} \right)$$

$$\frac{\Gamma\left(1-\frac{\epsilon}{2}+l\right)\Gamma(k+l+1)\Gamma(N-1-k-l)}{\Gamma(l+2)\Gamma\left(3-\frac{\epsilon}{2}+k+l\right)} \sum_{\sigma=0}^{\infty} \frac{(1-\epsilon)_\sigma \left(1+\frac{\epsilon}{2}\right)_\sigma (k+l+2)_\sigma}{(k+l+2)_\sigma \left(3-\frac{\epsilon}{2}+k+l\right)_\sigma \Gamma(\sigma+1)}$$

What tools do we have?

Wegschaider's Algorithm[1] - an extension of Fasenmyer/WZ summation - computes recurrences in elements of $\mu = (\mu_1, \dots, \mu_l)$ for the multiple sum:

$$Sum(\mu) := \sum_{\kappa_1} \cdots \sum_{\kappa_r} \mathcal{F}\left(\mu, \kappa_1, \dots, \kappa_r\right)$$

where $\mathcal{F}\left(\mu, \kappa\right)$ is a hypergeometric term in the variables $\mu = (\mu_1, \dots, \mu_l)$ and $\kappa = (\kappa_1, \dots, \kappa_r)$.

Remark: for $r = 1$ and $l = 1$ we have Zeilberger's Algorithm

[1] implemented in the MMA package `MultiSum` which is available at `http://www.risc.uni-linz.ac.at/research/combinat/software/`

Wegschaider's Algorithm more precisely

For the summation problem

$$Sum(\mu) := \sum_{\kappa_1} \cdots \sum_{\kappa_r} \mathcal{F}(\mu, \kappa_1, \ldots, \kappa_r)$$

MultiSum delivers a certificate recurrence of the form

$$\sum_{m \in \mathbb{S}} a_m(\mu) \mathcal{F}(\mu - m, \kappa) = \sum_{i=1}^{r} \Delta_{\kappa_i} \left(\sum_{(j,k) \in \mathbb{S}_i} p_{j,k}(\mu, \kappa) \mathcal{F}(\mu - j, \kappa - k) \right)$$

with $a_m(\mu)$ and $p_{j,k}(\mu, \kappa)$ polynomials.

Since $\mathcal{F}(\mu, \kappa)$ does not have finite support, we get a inhomogeneous recurrence for the sum

$$\sum_{m \in \mathbb{S}} a_m(\mu) \, Sum(\mu - m) = RHS$$

Remark:

$$\Delta_{\kappa_i} \mathcal{F}(\mu, \kappa) := \mathcal{F}(\mu, \kappa_1, \ldots, \kappa_i + 1, \ldots, \kappa_r) - \mathcal{F}(\mu, \kappa)$$

Unfolding recurrences

Given the first order inhomogeneous recurrence

$$Sum(n+1) - c(n)Sum(n) = h(n)$$

one can write

$$\begin{aligned}
Sum(n+1) =& c(n)Sum(n) + h(n) \\
=& c(n)\left(c(n-1)Sum(n-1) + h(n-1)\right) + h(n) \\
=& \dots
\end{aligned}$$

$$Sum(n+1) = \left(\prod_{i=1}^{n} c(i)\right) Sum(1) + \sum_{i=1}^{n-1} h(n-i) \prod_{j=0}^{i} c(j) + h(n)$$

go to Sigma

Higher order recurrences

$$Sum(n+2) - \alpha(n)Sum(n+1) - \beta(n)Sum(n) = h(n)$$

can be rewritten using operator notation

$$\left(S_n^2 - \alpha(n)S_n - \beta(n)\mathcal{I}\right) SUM(n) = h(n)$$

Using Hyper[2] we find a factorization of the form

$$\left(S_n - c(n)\mathcal{I}\right)\left(S_n - d(n)\mathcal{I}\right) Sum(n) = h(n)$$

if it exists.

[2] M. Petkovsek, *Hypergeometric solutions of linear recurrences with polynomial oefficients*, J. Symb. Comp. 14(1992)

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back 1st order recurrences

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ΠΣ extensions[3]

$$\sum_{i=1}^{n} iH_i = \sum_{i=1}^{n} i \sum_{j=1}^{i} \frac{1}{j} =?$$

$$K(i, H_i) \quad \sigma(H_i) = H_i + \tfrac{1}{i+1}$$

$$|$$

$$K(i) \qquad \sigma(i) = i + 1$$

$$|$$

$$K = \mathbb{Q}$$

Solve in the field $(K(i, H_i), \sigma)$ the difference equation:

$$\boxed{\sigma(g) - g = i \cdot H_i}$$

Sigma delivers the solution:

$$g = \frac{1}{4} \cdot i \cdot (i-1) \cdot (2H_i - 1)$$

$$\sum_{i=1}^{n} iH_i = g(n+1) - g(1) = \frac{1}{4} \cdot n \cdot (n-1) \cdot (2H_n - 1)$$

[3] C. Schneider, *Symbolic Summation Assists Combinatorics*, Sem. Lothar.
http://www.risc.uni-linz.ac.at/research/combinat/publications/

Setting up the inhomogeneous recurrence

E.g. $\boxed{Sum(N) = \sum_{\sigma=0}^{\infty} \sum_{k=0}^{N} \sum_{l=0}^{k} F(N,\sigma,k,l)}$ and MultiSum delivers

$$F(N+1,\sigma,k,l) + (N+2)F(N,\sigma,k,l) =$$

$$= \Delta_k \left[(l-k+1)F(N+1,\sigma,k,l) + F(N,\sigma,k,l)\right] \quad \Bigg| \sum_{\sigma=0}^{\infty} \sum_{k=0}^{N} \sum_{l=0}^{k}$$

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$$\sum_{\sigma=0}^{\infty} \sum_{k=0}^{N} \sum_{l=0}^{k} F(N+1, \sigma, k, l) = Sum(N+1) - \underbrace{\sum_{\sigma=0}^{\infty} \sum_{l=0}^{N+1} F(N+1, \sigma, N+1, l)}_{\text{shift compensating sum}}$$

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$$\sum_{\sigma=0}^{\infty} \sum_{l=0}^{k} \Delta_k \left[(l-k+1)F(N+1,\sigma,k,l)\right] \Big|_{k=0}^{k=N+1} \quad =$$

$$= \underbrace{\sum_{\sigma=0}^{\infty} \sum_{l=0}^{k} (l-k+1)F(N+1,\sigma,k,l) \Big|_{k=0}^{k=N+1}}_{\Delta\text{-boundary sums}} - \underbrace{\sum_{\sigma=0}^{\infty} \sum_{k=1}^{N+1} F(N+1,\sigma,k,k)}_{\text{telescoping compensation}}$$

$$\sum_{\sigma=0}^{\infty}\sum_{l=0}^{k}\Delta_k\left[F(N,\sigma,k,l)\right]\Bigg|_{k=0}^{k=N+1} =$$

$$= \sum_{\sigma=0}^{\infty}\sum_{l=0}^{k}F(N,\sigma,k,l)\Bigg|_{k=0}^{k=N+1} - \sum_{\sigma=0}^{\infty}\sum_{k=1}^{N+1}F(N,\sigma,k,k)$$

The inhomogeneous recurrence:

$$Sum(N+1)+(N+2)Sum(N)=\sum_{\sigma=0}^{\infty}\sum_{l=0}^{N+1}F(N+1,\sigma,N+1,l)$$

$$+\sum_{\sigma=0}^{\infty}\sum_{l=0}^{k}(l-k+1)F(N+1,\sigma,k,l)\Big|_{k=0}^{k=N+1}-\sum_{\sigma=0}^{\infty}\sum_{k=1}^{N+1}F(N+1,\sigma,k,k)$$

$$+\sum_{\sigma=0}^{\infty}\sum_{l=0}^{k}F(N,\sigma,k,l)\Big|_{k=0}^{k=N+1}-\sum_{\sigma=0}^{\infty}\sum_{k=1}^{N+1}F(N,\sigma,k,k)$$

Treating the "sore spot" separately:

$$Sum(N) = \sum_{\sigma=0}^{\infty} \sum_{k=0}^{N} \sum_{l=0}^{k} F(N, \sigma, k, l)$$

$$= \underbrace{\sum_{\sigma=0}^{\infty} \sum_{k=0}^{N-1} \sum_{l=0}^{k} F(N, \sigma, k, l)}_{Sum'(N)} + \underbrace{\sum_{\sigma=0}^{\infty} \sum_{l=0}^{N} F(N, \sigma, N, l)}_{\text{sore spot at } k=N}$$

The shift strategy:

Shifting in N the certificate recurrence

$$F(N+1,\sigma,k,l)+(N+2)F(N,\sigma,k,l)=$$

$$=\Delta_k\left[(l-k+1)F(N+1,\sigma,k,l)+F(N,\sigma,k,l)\right]$$

we obtain

$$F(N+2,\sigma,k,l)+(N+3)F(N+1,\sigma,k,l)=$$

$$=\Delta_k\left[(l-k+1)F(N+2,\sigma,k,l)+F(N+1,\sigma,k,l)\right] \quad \Bigg|\sum_{\sigma=0}^{\infty}\sum_{k=0}^{N}\sum_{l=0}^{k}$$

Future work

- recurrences for Mellin-Barnes representations of Feynman integrals[4]
- Sage implementation of Sigma and MultiSum coming soon

[4] F.S., *On recurrences for Ising integrals*, Adv. in Appl. Math., to appear