ISOLDE – A Maple package for systems of linear functional equations

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Introduction

ISOLDE implements algorithms for systems of linear differential, difference and q-difference equations:

- *Local problems:*
  compute formal invariants and solutions at a given point
- *Global problems:*
  compute closed form solutions (polynomial, rational, exponential).

Developed by Moulay Barkatou and Eckhard Pflügel since the 90's `http://isolde.sourceforge.net`

Preview:

A system of *pseudo-linear* equations over (F, ϕ, δ) has the form

$$\delta(Y) = A\,\phi(Y)$$

where $A \in \mathcal{M}_{n\times n}(F)$ and $Y \in F^n$ a column vector.

Discrete valuation rings

- F a discrete valuation field of characteristic zero. A map $v : F \to \mathbb{Z} \cup \{+\infty\}$ is a valuation on F if
  - $v(f) = +\infty \Leftrightarrow f = 0$
  - $v(fg) = v(f) + v(g)$
  - $v(f+g) \geq \min(v(f), v(g))$
- $\mathcal{O} = \{f \in F : v(f) \geq 0\}$ the valuation ring of F and $\mathcal{M} = \{f \in F : v(f) > 0\}$ its unique maximal ideal
- $\overline{F} = \mathcal{O}/\mathcal{M}$ is the residue field of F
- A local parameter $t \in \mathcal{O}$ has the property that $v(t) = 1$.
- This leads to a setting where every $f \in F$ can be expanded in a unique way w.r.t. a local parameter t

$$f = \sum_{i \geq \nu} f_i t^i$$

where $\nu = v(f)$, $f_\nu \neq 0$ and $f_i \in \overline{F}$.

Example

Let K a field with $char(K) = 0$ and $\mathbb{Q} \subset K$.
We regard $F = K((x))$ equipped with the x-adic valuation.
If

$$f = \sum_i f_i x^i \in F^*$$

then

$$v(f) = \inf\{i \text{ such that } f_i \neq 0\}.$$

- $\mathcal{O} = K[[x]]$ the ring of formal power series in x over K.
- The residue field of F is K.
- Similarly we can define a t-adic valuation where $t \in K[x]$ irreducible is a local parameter of F.

Example

For $F = K((x^{-1}))$ equipped with the x^{-1}-adic valuation we have

$$f = \sum_i f_i x^{-i} \in F^*$$

with

$$v(f) = \inf\{i \text{ such that } f_i \neq 0\}.$$

- The valuation ring of F is $\mathcal{O} = K[[x^{-1}]]$.
- The residue field of F is K.

Pseudo-derivations on discrete valuation fields

- ϕ an *isometry*, i.e. an automorphism of F s.t. $v(\phi f) = v(f)$ for all $f \in F$.
- $\delta : F \to F$ a *pseudo-derivation* or ϕ-derivation
  For all $a, b \in F$

$$\delta(a+b) = \delta a + \delta b \quad \text{and} \quad \delta(ab) = \phi(a)\delta b + \delta a b$$

- We define
  - $\omega \in \mathbb{Z}$ by $\omega(\delta) = v(t^{-1}\delta(t))$ for a continuous $\delta \neq 0$
  - $q \in \overline{F}$ by $\phi(t) = qt + O(t^2)$

  where t is a local parameter.

- If $\phi \neq 1_F$ then δ is of the form $\delta = \gamma(1_F - \phi)$ for some $\gamma \in F$. Otherwise, $\delta = \frac{d}{dt}$ is the usual derivation.

Systems of Pseudo-Linear Equations

A system of *pseudo-linear* equations over (F, ϕ, δ) has the form

$$_{\delta,\phi}[A] \qquad\qquad \delta(Y) = A\,\phi(Y)$$

where $A \in \mathcal{M}_{n\times n}(F)$ and $Y \in F^n$ a column vector.

Remark:

For a matrix $A = (a_{ij})$ we define

$$v(A) := \min v(a_{ij})$$

and we have expansions of the form

$$A = t^{v(A)} \sum_{i\geq 0} A_i t^i$$

with $A \in \mathcal{M}_{n\times n}(\overline{F})$, $A_0 \neq 0$.

In this framework

- Linear differential systems:

$$\phi = \mathrm{id},\ \delta = \frac{d}{dt},\ t = x,\ \omega = -1,\ q = 1$$

- Linear difference systems:

$$\phi(t) = \frac{t}{t+1},\ \delta = \phi - \mathrm{id},\ t = x^{-1},\ \ \omega = 1,\ q = 1$$

- Linear *q*-difference systems:

$$\phi(t) = qt,\ \delta = \phi - \mathrm{id},\ t = x^{-1},\ \omega = 0,\ q \neq 0$$

Formal reduction

Any differential system $_{\delta,\phi}[A]$ is equivalent over $\overline{F}$ to a system $_{\delta,\phi}[B]$ in triangular form (the Turritin normal form). Their exponential parts are given by:

$$w_i \sim \Gamma_1^{(i)} x^{-k_1-1} + \Gamma_2^{(i)} x^{-k_2-1} + \cdots + \Gamma_m^{(i)} x^{-k_m-1} + \Gamma_0^{(i)} x^{-1}$$

where $k_1 > k_2 > \cdots > k_m > 0$ are rational numbers and the Γ's form constant diagonal matrices

$$\Gamma_i := diag\left(\Gamma_i^{(1)}, \ldots, \Gamma_i^{(n)}\right).$$

The building blocks of the FormalReduction procedure

- Pseudo-linear Transformations
- Moser-/Super-reduction
- Splitting Lemma
- Generalised Splitting Lemma
- Ramifications

Analysis of singularities

Given $T \in GL_n(F)$, the change of variable $Y = TZ$ yields the equivalent system

$$ {}_{\delta,\phi}[B] \qquad\qquad \delta(Z) = B\,\phi(Z) $$

where $B = T_{\delta,\phi}[A] := T^{-1}A\phi(T) - T^{-1}\delta(T)$.

Definition:

A pseudo-linear system ${}_{\delta,\phi}[A]$ is called regular if there exists a gauge transformation $T \in GL_n(F)$ such that $v(T_{\delta,\phi}[A]) \geq \omega(\delta)$.

Problem:

Given a system ${}_{\delta,\phi}[A]$ decide whether it is regular singular or compute a gauge transformation T s.t. the valuation of $T_{\delta,\phi}[A]$ is as close as possible to $\omega(\delta)$.

[Moser '60, Hilali-Wazner '87, Barkatou '95, Barkatou-Broughton-Pflügel '08]

Splitting lemma

Let $\;_{\delta,\phi}[A]$ be a reduced system in expanded form

$$A = t^{v(A)} \sum_{i \geq} A_i t^i$$

with $A_i \in \mathcal{M}_{n\times n}(\overline{F})$, $A_0 \neq 0$ and

$$A_0 = \begin{pmatrix} A_0^{11} & 0 \\ 0 & A_0^{22} \end{pmatrix}.$$

such that $spec(A_0^{11}) \cap spec(A_0^{22}) = \emptyset$.
Then we can compute $T \in GL_n$ with $T_0 = I_n$ s.t. the equivalent system is

$$T_{\delta,\phi}[A] = B = \begin{pmatrix} B^{11} & 0 \\ 0 & B^{22} \end{pmatrix}.$$

with block sizes matching the block structure of A_0.
[Wasow '67, Barkatou '93, Pflügel '00, Barkatou-Broughton-Pflügel '10]

The algorithm

1. $p := p(A)$; $n := dim(A)$; A_0 the leading matrix of $\;_{\delta,\phi}[A]$;
2. if A_0 nilpotent then make $[A]$ Moser-reduced (p minimal)
3. if $n = 1$ or $p = 0$ then STOP

4. if $p > \omega(\delta)$ and $n > 1$ and A_0 nilpotent then

4.1 compute $\kappa(A) = \frac{e}{m}$ with $gcd(e, m) = 1$
4.2 replace t by t^m
4.3 Make $\;_{\delta,\phi}[A]$ Moser-reduced

5. if A_0 is not nilpotent then for each eigenvalue $\lambda \neq 0$ do

5.1 $p := p(A)$; $A := A - \frac{\lambda}{t^{p+1}} I_n$;
5.2 Splitting gives $B := diag(B^{(1)}, B^{(2)})$ with $B_0^{(2)}$ nilpotent.
5.3 $A := B^{(2)}$ and go to Step 1

- At each step, one either reduces the Poincaré rank p or the order n of the system.

[Barkatou '97, Barkatou-Pflügel '00]

Conclusions

The ISOLDE package now adopts a unifying view for systems of differential, difference and q-difference equations.

- generic algorithms that can be applied to individual systems by giving specific input parameters