Symbolic summation for Feynman integrals

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A Feynman parameter integral

$$I = \int_0^1 \int_0^1 \frac{(1-w)^{-1-\epsilon/2} z^{\epsilon/2} (1-z)^{-\epsilon/2}}{(1-wz)^{1-\epsilon}} (1-w)^{N+1} \, dw \, dz,$$

where $N \in \mathbb{N}$ and $\epsilon > 0$.

- Symbolic summation methods
- Feynman integrals as multiple sums
- Feynman integrals as Mellin-Barnes integrals

Part 1: Symbolic summation methods

Hypergeometric series

A simple example

$$ {}_3F_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix} ; x\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c)_n}{(d)_n (e)_n} \frac{x^n}{n!} $$

$$ = \frac{\Gamma(d)\Gamma(e)}{\Gamma(b)\Gamma(c)\Gamma(d-b)\Gamma(e-c)} \int_0^1 \int_0^1 \frac{t^{b-1}(1-t)^{d-b-1} u^{c-1}(1-u)^{e-c-1}}{(1-utx)^a} dt\, du $$

$$ {}_3F_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}\; ; x\right) = \sum_{n=0}^{\infty} \overbrace{\frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n}\frac{x^n}{n!}}^{t_n} $$

where the Pochhammer symbol / rising factorial is

$$ (a)_n := a(a+1)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(n)} $$

if $a \in \mathbb{C}$ and $a + n \neq 0, -1, -2, \dots$

$$ \frac{t_{n+1}}{t_n} = \frac{(a+n)(b+n)(c+n)}{(d+n)(e+n)}\frac{x}{n+1} $$

An illustrative application for summation methods[1]

For $n \geq k \geq 1$ prove the following identity:

$$1+(-1)^{n+k}\sum_{m\geq 0}(-1)^m\binom{k-1-n}{m}\binom{n}{k-1-m} = 2^k\sum_{m\geq k}\binom{-k}{n-m}$$

[1] question asked by P. van der Kamp

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Initial values:

- if $n = 1$ then $k = 1$
- if $n = 2$ then $k \in \{1, 2\}$

[1] question asked by P. van der Kamp

Zeilberger's Algorithm[2,3]

Given the summation problem

$$\texttt{SUM[n]} := \sum_{m=b}^{B} f(m,n)$$

Zeilberger's Algorithm delivers certificate recurrences of the form

$$\sum_{i \in \mathbb{S}} c_i(n) f(m, n+i) = \Delta_m\,[g(m,n)]$$

where $\quad g(m,n) = rat(m,n) f(m,n) \quad$ and

$$\Delta_m\,[g(m,n)] := g(m+1,n) - g(m,n).$$

[2] D. Zeilberger *A fast algorithm for proving terminating hypergeometric identities.* Discrete Mathematics, 1990.

[3] P. Paule, M. Schorn *A Mathematica version of Zeilberger's algorithm for proving binomial coefficient identities.* J. Symbolic Comput. 1995.

Summing over the certificate recurrences

$$\sum_{i\in\mathbb{S}} c_i(n) f(m, n+i) = \Delta_m\left[g(m, n)\right]$$

we obtain recurrences for the sum $\texttt{SUM[n]} := \sum_{m=b}^{B} f(m, n)$

$$\sum_{i\in\mathbb{S}} c_i(n)\texttt{SUM[n+i]} = g(B+1, n) - g(b, n).$$

Remark: Zeilberger's algorithm also deals with definite summation problems $\rightarrow$ the bounds b, B can lineary depend on n.

Back to the illustrative example

$$1+(-1)^{k+n}\sum_{m=0}^{k-1}(-1)^m\binom{k-1-n}{m}\binom{n}{k-1-m} = 2^k \underbrace{\sum_{m=k}^{n}\binom{-k}{n-m}}_{T(n)}$$

We extend the sum $T(n)$ to a summation problem with standard boundary conditions:

$$T(n) := \sum_{m=k}^{n}\binom{-k}{n-m} = \lim_{\epsilon\to 0}\underbrace{\sum_{m=-\infty}^{\infty}\binom{-k}{n-m}\frac{(m-k+\epsilon)!}{(m-k)!}}_{t(n,\epsilon)}$$

Zeilberger's algorithm delivers a recurrence for $t(n,\epsilon)$

$$(\epsilon+n+1)t(n,\epsilon)+(\epsilon-k+1)t(n+1,\epsilon)+(k-n-2)t(n+2,\epsilon)=0$$

Taking here $\epsilon \to 0$, we come to the conclusion that $T(n)$ satisfies

$$(n+1)T(n)+(1-k)T(n+1)+(k-n-2)T(n+2)=0$$

while for the LHS of the identity we computed the same recurrence

$$(n+1)\texttt{SUM[n]}+(1-k)\texttt{SUM[n+1]}+(k-n-2)\texttt{SUM[n+2]}=0.$$

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A multisum example[5]

For $r \geq k \geq 0$, $s \geq 0$ prove that:

$$\sum_{i=0}^{r-k}\sum_{j=0}^{s}\binom{r-j}{i}\binom{r+k-i}{j}\binom{k+i}{s-j}\binom{r-s+j}{r-k-i} =$$

$$=\sum_{i=0}^{r-k}\sum_{j=0}^{s}\binom{r-j+2}{i}\binom{r+k-i}{j}\binom{k+i}{s-j}\binom{r-s+j-2}{r-k-i}$$

MultiSum[4] delivers the same recurrence for both sides:

$$(2k+r+2)(2r-s+2)\texttt{SUM[s,r]} - (2r^2+4kr+8r+2k-3ks+8)$$
$$\texttt{SUM[s+1,r+1]} - (k-r-2)(s+2)\texttt{SUM[s+2,r+2]} = 0$$

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[4] K. Wegschaider, *Computer generated proofs of binomial multi-sum identities*, Diploma thesis, RISC, Johannes Kepler University, 1997

[5] question asked by H. Markwig

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$$= \sum_{i=0}^{r-k}\sum_{j=0}^{s}\binom{r-j+2}{i}\binom{r+k-i}{j}\binom{k+i}{s-j}\binom{r-s+j-2}{r-k-i}$$

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More general: WZ-summation methods

WZ-summation methods deliver recurrences in elements of $\mu = (\mu_1, \ldots, \mu_l)$ for the multiple sum:

$$Sum(\mu) := \sum_{\kappa_1} \cdots \sum_{\kappa_r} \mathcal{F}(\mu, \kappa_1, \ldots, \kappa_r)$$

where $\mathcal{F}(\mu, \kappa)$ is a hypergeometric term in the variables $\mu = (\mu_1, \ldots, \mu_l)$ and $\kappa = (\kappa_1, \ldots, \kappa_r)$.

Remark: for $r = 1$ and $l = 1$ we have Zeilberger's Algorithm

Given a κ-free recurrence[6] satisfied by the summand

$$\sum_{(u,v)\in\mathbb{S}} a_{u,v}(\mu,\alpha)\,\mathcal{F}(\mu+u,\kappa+v)=0$$

one computes a certificate recurrence[7]

$$\sum_{m\in\mathbb{S}'} c_m(\mu)\,\mathcal{F}(\mu+m,\kappa) = \sum_{i=1}^{r}\Delta_{\kappa_i}\left(\sum_{(m,k)\in\mathbb{S}_i} p_{j,k}(\mu,\kappa,)\,\mathcal{F}(\mu+j,\kappa+k)\right),$$

where

$$\Delta_{\kappa_i}\mathcal{F}(\mu,\kappa) := \mathcal{F}(\mu,\kappa_1,\ldots,\kappa_i+1,\ldots,\kappa_r)-\mathcal{F}(\mu,\kappa)\,.$$

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[6] M.C. Fasenmyer, *Some generalized hypergeometric polynomials*, PhD thesis, Univ. of Michigan, 1945

[7] H.S. Wilf and D. Zeilberger, *An algorithmic proof theory for hypergeometric (ordinary and q) multisum/integral identities*, Inventiones Mathematicae, 1992

Summary of Part 1

For the summation problem

$$Sum(\mu) := \sum_{\kappa_1} \cdots \sum_{\kappa_r} \mathcal{F}(\mu, \kappa_1, \ldots, \kappa_r)$$

WZ-methods give a certificate recurrence of the form

$$\sum_{m \in \mathbb{S}} c_m(\mu)\, \mathcal{F}(\mu + m, \kappa) = \sum_{i=1}^{r} \Delta_{\kappa_i} \left(\sum_{(j,k) \in \mathbb{S}_i} p_{j,k}(\mu, \kappa)\, \mathcal{F}(\mu + j, \kappa + k) \right)$$

By summing over the certificate recurrence we obtain a recurrence for the sum

$$\sum_{m \in \mathbb{S}} c_m(\mu)\, Sum(\mu + m) = RHS$$

Remark:

$$\Delta_{\kappa_i} \mathcal{F}(\mu, \kappa) := \mathcal{F}(\mu, \kappa_1, \ldots, \kappa_i + 1, \ldots, \kappa_r) - \mathcal{F}(\mu, \kappa)$$

Part 2: Feynman integrals as multiple sums [8]

[8] joint work with Carsten Schneider (RISC) and Johannes Blümlein (DESY)

Back to the Feynman integral example

$$I = \int_0^1 \int_0^1 \frac{(1-w)^{-1-\epsilon/2} z^{\epsilon/2} (1-z)^{-\epsilon/2}}{(1-wz)^{1-\epsilon}} (1-w)^{N+1}\, dw\, dz,$$

where $N \in \mathbb{N}$ and $\epsilon > 0$.

.. seen as a hypergeometric series

$$I = \int_0^1 \int_0^1 \frac{(1-w)^{N-\epsilon/2} z^{\epsilon/2} (1-z)^{-\epsilon/2}}{(1-wz)^{1-\epsilon}} \, dw \, dz$$

$$= \frac{\Gamma\left(1-\frac{\epsilon}{2}\right)\Gamma\left(1+\frac{\epsilon}{2}\right)}{2\left(N+1-\frac{\epsilon}{2}\right)} \underbrace{{}_3F_2\left(\begin{matrix} 1, 1-\epsilon, 1+\frac{\epsilon}{2} \\ 2, N+2-\frac{\epsilon}{2} \end{matrix}; 1\right)}_{\underbrace{\sum_{\sigma \geq 0} \frac{(1-\epsilon)_\sigma \left(1+\frac{\epsilon}{2}\right)_\sigma}{(2)_\sigma \left(N+2-\frac{\epsilon}{2}\right)_\sigma}}_{\mathtt{SUM[N]}}}$$

MultiSum[9] computes a certificate recurrence satisfied by its summand

Out[0]=

$$\textit{certRec} = (N+1)(2N-\epsilon+2)F[N,\sigma]-(N-\epsilon+1)(2N+\epsilon+2)F[N+1,\sigma] = \Delta_\sigma[(-\sigma-1)(2N-\epsilon+2)F[N,\sigma]]$$

The sum representation of I, denoted by `SUM[N]`, satisfies the recurrence

Out[0]=

$$\textit{rec} = (N+1)(2N-\epsilon+2)\texttt{SUM[N]} \\ -(N-\epsilon+1)(2N+\epsilon+2)\texttt{SUM[N+1]} == \Gamma\left(1-\tfrac{\epsilon}{2}\right)\Gamma\left(\tfrac{\epsilon}{2}+1\right).$$

[9]K. Wegschaider, *Computer generated proofs of binomial multi-sum identities*, Diploma thesis, RISC, Johannes Kepler University, 1997

Unfolding recurrences

Given the first order inhomogeneous recurrence

$$\texttt{SUM[N+1]} - c(n)\texttt{SUM[N]} = h(n)$$

one can write

$$\begin{aligned}\texttt{SUM[N+1]} &= c(n)\texttt{SUM[N]} + h(n)\\ &= c(n)\left(c(n-1)\texttt{SUM[N-1]} + h(n-1)\right) + h(n)\\ &= \ldots\end{aligned}$$

$$\texttt{SUM[N+1]} = \left(\prod_{i=1}^{n} c(i)\right)\texttt{SUM[1]} + \sum_{i=1}^{n-1} h(n-i)\prod_{j=0}^{i} c(j) + h(n)$$

ΠΣ extensions [10]

$$\sum_{i=1}^{n} iH_i = \sum_{i=1}^{n} i \sum_{j=1}^{i} \frac{1}{j} = ?$$

$$\begin{array}{ll} K(i,H_i) & \sigma(H_i) = H_i + \frac{1}{i+1} \\ \quad | & \\ K(i) & \sigma(i) = i+1 \\ \quad | & \\ K = \mathbb{Q} & \end{array}$$

Solve in the field $K(i,H_i),\sigma$ the difference equation:

$$\boxed{\Delta(g) = \sigma(g) - g = i \cdot H_i}$$

`Sigma` delivers the solution:

$$g = \frac{i(i-1)}{4} \cdot (2H_i - 1)$$

$$\sum_{i=1}^{n} iH_i = g(n+1) - g(1) = \frac{n(n-1)}{4} \cdot (2H_n - 1)$$

[10] C. Schneider *Symbolic Summation Assists Combinatorics*, Sém. Lothar. Combin. 2007

Higher order recurrences

$$\texttt{SUM[n+2]} - \alpha(n)\texttt{SUM[n+1]} - \beta(n)\texttt{SUM[n]} = h(n)$$

can be rewritten using operator notation

$$\left(S_n^2 - \alpha(n)S_n - \beta(n)\mathcal{I}\right)\texttt{SUM[n]} = h(n)$$

Using Hyper[11] we find a factorization of the form

$$(S_n - c(n)\mathcal{I})\,(S_n - d(n)\mathcal{I})\texttt{SUM[n]} = h(n)$$

if it exists.

[11] M. Petkovsek *Hypergeometric solutions of linear recurrences with polynomial coefficients* J. Symbolic Comp. 1992

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[11] M. Petkovsek *Hypergeometric solutions of linear recurrences with polynomial coefficients* J. Symbolic Comp. 1992

A realistic example

$$\mathcal{U}(N,\epsilon) := (-1)^N \sum_{\sigma_0=0}^{\infty} \sum_{j_0=0}^{N-3} \sum_{j_1=0}^{N-j_0-3} \sum_{j_2=0}^{j_0+1} \binom{j_0+1}{j_2}\binom{N-j_0-3}{j_1}$$
$$\times \frac{\left(\frac{\epsilon}{2}+1\right)_{\sigma_0}(-\epsilon)_{\sigma_0}(j_1+j_2+3)_{\sigma_0}\left(3-\frac{\epsilon}{2}\right)_{j_1}}{(j_1+4)_{\sigma_0}\left(-\frac{\epsilon}{2}+j_1+j_2+4\right)_{\sigma_0}\left(4-\frac{\epsilon}{2}\right)_{j_1+j_2}}$$
$$\times \frac{\Gamma(j_1+j_2+2)\Gamma(j_1+j_2+3)\Gamma(N-j_0-1)\Gamma(N-j_1-j_2-1)}{\Gamma(\sigma_0+1)\Gamma(j_1+4)\Gamma(N-j_0-2)}$$

where $N \geq 3$ is the Mellin moment and $\epsilon > 0$ is the dimension regularization parameter.

Setting up the inhomogeneous recurrence

E.g. $\boxed{\mathtt{SUM}[N] = \sum_{\sigma=0}^{\infty} \sum_{j_0=0}^{N} \sum_{j_1=0}^{j_0} \mathtt{F}[N, \sigma, j_0, j_1]}$ and `MultiSum` delivers

the certificate recurrence

$$\mathtt{F}[N+1, \sigma, j_0, j_1] + (N+2)\mathtt{F}[N, \sigma, j_0, j_1] =$$

$$= \Delta_{j_0}\left[(j_1 - j_0 + 1)\mathtt{F}[N+1, \sigma, j_0, j_1] + \mathtt{F}[N, \sigma, j_0, j_1]\right] \quad \Bigg| \sum_{\sigma=0}^{\infty} \sum_{j_0=0}^{N} \sum_{j_1=0}^{j_0}$$

$$\sum_{\sigma=0}^{\infty}\sum_{j_0=0}^{N}\sum_{j_1=0}^{j_0} F(N+1,\sigma,j_0,j_1) = \mathrm{SUM}[N+1] \underbrace{- \sum_{\sigma=0}^{\infty}\sum_{j_1=0}^{N+1} F(N+1,\sigma,N+1,j_1)}_{\text{shift compensation}}$$

$$\sum_{\sigma=0}^{\infty}\sum_{j_1=0}^{j_0} \Delta_{j_0}\left[(j_1 - j_0 + 1)F(N+1,\sigma,j_0,j_1)\right]\Bigg|_{k=0}^{j_0=N+1} =$$

$$= \underbrace{\sum_{\sigma=0}^{\infty}\sum_{j_1=0}^{j_0}(j_1 - j_0 + 1)F(N+1,\sigma,j_0,j_1)\Bigg|_{j_0=0}^{j_0=N+1}}_{\Delta\text{-boundary sums}}$$

$$- \underbrace{\sum_{\sigma=0}^{\infty}\sum_{j_0=1}^{N+1} F(N+1,\sigma,j_0,j_0)}_{\text{telescoping compensation}}$$

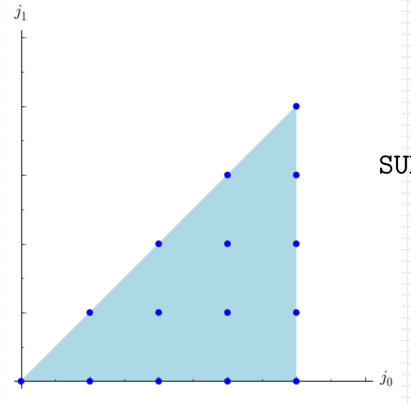
The inhomogeneous recurrence

$$\mathtt{SUM}[N+1] + (N+2)\mathtt{SUM}[N] = \sum_{\sigma=0}^{\infty}\sum_{j_1=0}^{N+1} F[N+1,\sigma,N+1,j_1]$$

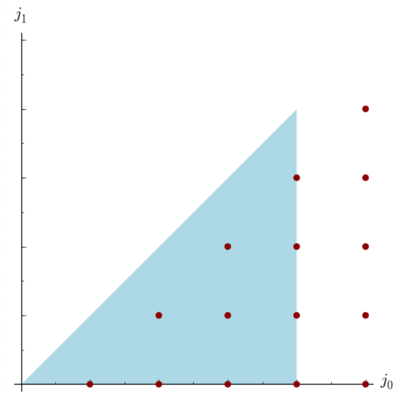
$$+\sum_{\sigma=0}^{\infty}\sum_{j_1=0}^{j_0}(j_1-j_0+1)F[N+1,\sigma,j_0,j_1]\Bigg|_{j_0=0}^{j_0=N+1} - \sum_{\sigma=0}^{\infty}\sum_{j_0=1}^{N+1} F[N,\sigma,j_0,j_0]$$

$$+\sum_{\sigma=0}^{\infty}\sum_{j_1=0}^{j_0} F[N,\sigma,j_0,j_1]\Bigg|_{j_0=0}^{j_0=N+1} - \sum_{\sigma=0}^{\infty}\sum_{j_0=1}^{N+1} F[N+1,\sigma,j_0,j_0]$$

where the RHS of the recurrence contains shift and telescoping compensating sums as well as delta boundary sums.

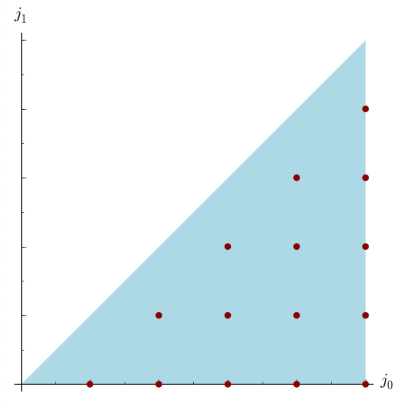


$$\mathtt{SUM}[\mathtt{N}] := \sum_{\sigma=0}^{\infty} \sum_{\mathtt{j_0}=0}^{\mathtt{N}} \sum_{\mathtt{j_1}=0}^{\mathtt{j_0}} \mathtt{F}[\mathtt{N}, \sigma, \mathtt{j_0}, \mathtt{j_1}]$$



$$\sum_{\sigma=0}^{\infty} \sum_{j_0=0}^{N} \sum_{j_1=0}^{j_0} F[N, \sigma, j_0 + 1, j_1]$$

$$= \sum_{\sigma=0}^{\infty} \sum_{j_0=1}^{N+1} \sum_{j_1=0}^{j_0-1} F[N, \sigma, j_0, j_1]$$



$$\sum_{\sigma=0}^{\infty}\sum_{j_0=0}^{N}\sum_{j_1=0}^{j_0} F[N+1,\sigma,j_0+1,j_1]$$

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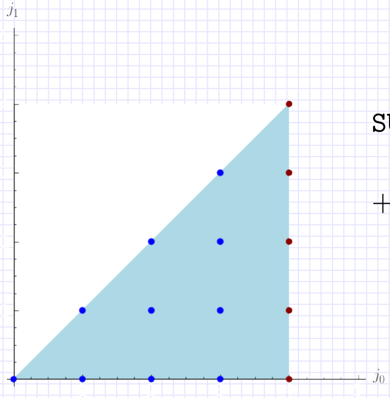


$$\mathtt{SUM}[\mathtt{N}] = \sum_{\sigma=0}^{\infty} \sum_{\mathtt{j}_0=0}^{\mathtt{N}-1} \sum_{\mathtt{j}_1=0}^{\mathtt{j}_0} \mathtt{F}[\mathtt{N}, \sigma, \mathtt{j}_0, \mathtt{j}_1]$$

$$+ \sum_{\sigma=0}^{\infty} \sum_{\mathtt{j}_1=0}^{\mathtt{N}} \mathtt{F}[\mathtt{N}, \sigma, \mathtt{N}, \mathtt{j}_1]$$

Part 3:

Feynman integrals as Mellin-Barnes integrals

The Mellin transform

The Mellin transform of a locally integrable function $f : (0, \infty) \to \mathbb{C}$ is

$$\tilde{f}(s) = \int_0^\infty x^{s-1} f(x) dx =: M[f; s]$$

defined usually on an infinite strip $a < Re(s) < b$.
The inversion formula

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \tilde{f}(s) ds$$

has a contour of integration placed in the strip of analyticity $a < c < b$.

For example

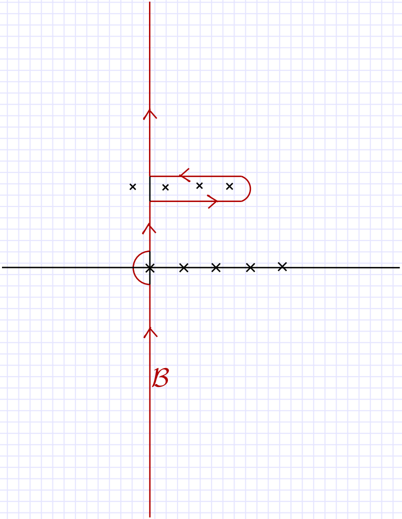
$$\frac{1}{(1-x)^a} = \frac{1}{2\pi i}\int_{\mathcal{B}} \frac{\Gamma(-s)\Gamma(a+s)}{\Gamma(a)}(-x)^s ds$$

where the contour $\mathcal{B}$ separates the set of ascending poles

$$\{0, 1, 2, 3, \dots\}$$

from the set of descending poles of the integrand

$$\{-a, -a-1, -a-2, \dots\}$$

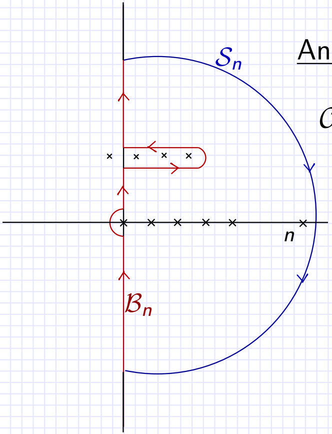


Let's ask the question

$$\int_{\mathcal{B}} \underbrace{\frac{\Gamma(a+s)\Gamma(-s)}{\Gamma(a)}(-z)^s}_{\Psi_{z,a}(s)} \, ds = ?$$

$\mathcal{B}$ is a Barnes contour of integration puting the poles of $\Gamma(a+s)$ to left and those of $\Gamma(-s)$ to the right

$$\int_{\mathcal{B}} \underbrace{\frac{\Gamma(a+s)\Gamma(-s)}{\Gamma(a)}(-z)^s}_{\Psi_{z,a}(s)} ds = ?$$



Answer:

$\mathcal{C}_n := \mathcal{B}_n \cup \mathcal{S}_n$ for all $n \geq 0$ and

$$\lim_{n\to\infty} \int_{\mathcal{S}_n} \Psi_{z,a}(s)ds = 0$$

then

$$\int_{\mathcal{B}} \Psi_{z,a}(s)ds = \lim_{n\to\infty} \underbrace{\int_{\mathcal{C}_n} \Psi_{z,a}(s)ds}_{(-2\pi i)\sum_{k=0}^{n} Res(\Psi(t),k)}$$

$$ {}_1F_0\left(\begin{array}{c} a \\ - \end{array}; z\right) = \frac{1}{(1-z)^a} = \frac{1}{2\pi i}\int_{\mathcal{B}} \frac{\Gamma(a+s)\Gamma(-s)}{\Gamma(a)}(-z)^s ds $$

More general,

$$ \frac{1}{2\pi i}\int_{\mathcal{B}} \frac{\prod_{l=1}^{p}\Gamma(a_l+s)\Gamma(-s)(-z)^s}{\prod_{j=1}^{q}\Gamma(b_j+s)} ds = $$

$$ = \frac{\prod_{l=1}^{p}\Gamma(a_l)}{\prod_{j=1}^{q}\Gamma(b_j)} \quad {}_pF_q\left(\begin{array}{c} a_1, a_2, \ldots, a_p \\ \\ b_1, b_2, \ldots, b_q \end{array}; z\right) $$

Back to our Feynman integral

$$I = \int_0^1 \int_0^1 \frac{(1-w)^{-1-\epsilon/2} z^{\epsilon/2} (1-z)^{-\epsilon/2}}{(1-wz)^{1-\epsilon}} (1-w)^{N+1}\, dw\, dz,$$

where $N \in \mathbb{N}$ and $\epsilon > 0$.

$$I = \int_0^1 dw \int_0^1 dz \quad (1-w)^{N-\epsilon/2} z^{\epsilon/2} (1-z)^{-\epsilon/2} \times \underbrace{\frac{1}{2\pi i} \int_{\mathcal{C}_s} \frac{\Gamma(-s)\Gamma(1-\epsilon+s)}{\Gamma(1-\epsilon)} (-wz)^s ds}_{1/(1-wz)^{1-\epsilon}}$$

$$= \frac{1}{2\pi i} \int_{\mathcal{C}_s} \frac{\Gamma(-s)\Gamma(1-\epsilon+s)}{\Gamma(1-\epsilon)} (-1)^s \times \underbrace{\left(\int_0^1 z^{\epsilon/2+s} (1-z)^{-\epsilon/2} dz \right)}_{B(\epsilon/2+s+1, -\epsilon/2+1)} \underbrace{\left(\int_0^1 w^s (1-w)^{N-\epsilon/2} dw \right)}_{B(s+1, N-\epsilon/2+1)} ds$$

$$I = \frac{\Gamma\left(1-\frac{\epsilon}{2}\right)\Gamma\left(N-\frac{\epsilon}{2}+1\right)}{4\pi i\Gamma(1-\epsilon)}$$

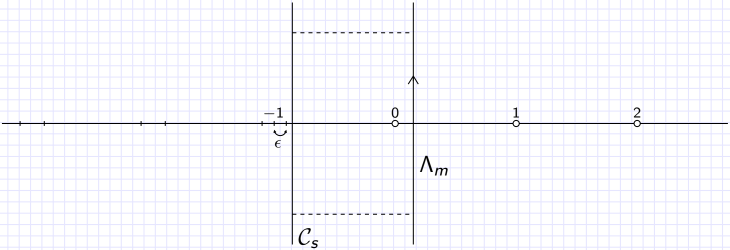
$$\times \int_{\mathcal{C}_s} \frac{\Gamma(s+1-\epsilon)\Gamma\left(s+1+\frac{\epsilon}{2}\right)}{(s+1)\Gamma\left(N+s-\frac{\epsilon}{2}+2\right)}\Gamma(-s)(-1)^s \, ds$$

MultiSum computes a certificate recurrence satisfied by its integrand

Out[0]=

$$certRec = (N+1)(2N-\epsilon+2)F[N,s]-(N-\epsilon+1)(2N+\epsilon+2)F[N+1,s] = \Delta_s[(-s-1)(2N-\epsilon+2)F[N,s]]$$

The contour $\mathcal{C}_s$ separates the ascending chain of poles of $\Gamma(-s)$ from the descending chains coming from $\frac{\Gamma(s+1-\epsilon)\Gamma\left(s+1+\frac{\epsilon}{2}\right)}{(s+1)}$.



On the RHS we have an improper integral

$$\int_{\mathcal{C}_s} \Delta_s[(-s-1)(2N-\epsilon+2)\mathtt{F[N,s]}] \quad ds$$

$$= -(2N-\epsilon+2)\lim_{m\to\infty}\int_{\Lambda_m}(s+1)\mathtt{F[N,s]}\, ds = \Gamma\left(1-\frac{\epsilon}{2}\right)\Gamma\left(\frac{\epsilon}{2}+1\right)$$

where F[N,s] denotes the integrand.

For the Feynman parameter integral

$$I = \int_0^1 \int_0^1 \frac{(1-w)^{-1-\epsilon/2} z^{\epsilon/2} (1-z)^{-\epsilon/2}}{(1-wz)^{1-\epsilon}} (1-w)^{N+1} \, dw \, dz,$$

the sum representation satisfies the recurrence

Out[0]= $rec = (N+1)(2N-\epsilon+2)\text{SUM}[N]$
$-(N-\epsilon+1)(2N+\epsilon+2)\text{SUM}[N+1] == \Gamma\left(1-\frac{\epsilon}{2}\right)\Gamma\left(\frac{\epsilon}{2}+1\right)$

and the Mellin-Barnes integral representation satisfies the same

In[1]:= **rec/.SUM → INT**

Out[1]= $(N+1)(2N-\epsilon+2)\text{INT}[N]$
$-(N-\epsilon+1)(2N+\epsilon+2)\text{INT}[N+1] == \Gamma\left(1-\frac{\epsilon}{2}\right)\Gamma\left(\frac{\epsilon}{2}+1\right).$

Recurrences for Mellin-Barnes integrals

For the integration problem

$$Int(\mu) := \int_{\mathcal{B}_{\kappa_1}} \dots \int_{\mathcal{B}_{\kappa_r}} \mathcal{F}(\mu, \kappa_1, \dots, \kappa_r) \; d\kappa$$

WZ-methods give a certificate recurrence of the form

$$\sum_{m \in \mathbb{S}} c_m(\mu) \mathcal{F}(\mu + m, \kappa) = \sum_{i=1}^{r} \Delta_{\kappa_i} \left(\sum_{(j,k) \in \mathbb{S}_i} p_{j,k}(\mu, \kappa) \mathcal{F}(\mu + j, \kappa + k) \right)$$

By integrating over the certificate recurrence we obtain a recurrence for the integral

$$\sum_{m \in \mathbb{S}} c_m(\mu) \, Int(\mu + m) = RHS$$

One more example

Identity **7.512.2** from the Table[12]

$$\int_0^1 x^{\rho-1}(1-x)^{\beta-\gamma-n}\,{}_2F_1\left(\begin{matrix}-n,\beta\\ \gamma\end{matrix};x\right)dx$$

$$=\frac{\Gamma(\gamma)\Gamma(\rho)\Gamma(\beta-\gamma+1)\Gamma(\gamma-\rho+n)}{\Gamma(\gamma+n)\Gamma(\gamma-\rho)\Gamma(\beta-\gamma+\rho+1)},$$

where $n \in \mathbb{N}$, $Re\,\rho > 0$ and $Re(\beta-\gamma) > n-1$.

[12] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series, and Products*

Recall the Mellin transform slide..

The Mellin transform of a locally integrable function $f : (0, \infty) \to \mathbb{C}$ is

$$\tilde{f}(s) = \int_0^{\infty} x^{s-1} f(x) dx =: M[f; s]$$

defined usually on an infinite strip $a < Re(s) < b$.

Remark: For a polynomial function $f(x) = (1 - x)^n$ we have

$$a = 0 \quad \text{and} \quad b = -n$$

$\Longrightarrow$ no strip of analyticity.

A generalized definition of the Mellin transform[13]

Decomposing $f(x)$ such that $f(x) = f_1(x) + f_2(x)$, e.g.,

$$f_1(x) = \begin{cases} f(x), & x \in [0,1) \\ 0, & x \in [1,\infty) \end{cases} \quad , \quad f_2(x) = \begin{cases} 0, & x \in [0,1) \\ f(x), & x \in [1,\infty) \end{cases}$$

we have

$$M[f;z] = M[f_1;z] + M[f_2;z].$$

In our case, $f(x) = (1-x)^n$ and its Mellin transform is

$$M[f;z] = \Gamma(n+1)\left[\frac{\Gamma(z)}{\Gamma(n+z+1)} + (-1)^n \frac{\Gamma(-n-z)}{\Gamma(1-z)}\right].$$

[13] N. Bleistein, R.A. Handelsman *Asymptotic Expansions of Integrals*

..we arrive at the Mellin-Barnes integral representation:

$$
{}_2F_1\left(\begin{matrix} -n, \beta \\ \gamma \end{matrix} ; x \right) =
$$

$$
= \frac{\Gamma(\gamma)\Gamma(n+1)}{\Gamma(\beta)} \frac{1}{2\pi i} \left[\int_{\delta - i\infty}^{\delta + i\infty} \frac{\Gamma(s)}{\Gamma(n+s+1)} \frac{\Gamma(\beta - s)}{\Gamma(\gamma - s)} x^{-s} ds \right.
$$

$$
\left. + (-1)^n \int_{\eta - i\infty}^{\eta + i\infty} \frac{\Gamma(-n-s)}{\Gamma(1-s)} \frac{\Gamma(\beta - s)}{\Gamma(\gamma - s)} x^{-s} ds \right]
$$

where $Re\,\beta > \delta > 0$ and $\eta < -n$.

Using the Mellin transform and WZ-methods we can now prove identities of the form[14]

$$\int_0^\infty e^{-x} x^{\alpha+\beta} L_m^\alpha(x) L_n^\beta(x) dx = (-1)^{m+n} (\alpha+\beta)! \binom{\alpha+m}{n} \binom{\beta+n}{m}$$

where $L_m^\alpha(x)$ are the Laguerre polynomials.

[14] Karen Kohl, FS. *An algorithmic approach to the Mellin transform method*

Conclusions

- real-world applications of symbolic summation methods
- hypergeometric summation with non-standard bounds
- computing recurrences for nested Mellin-Barnes integrals